- polar ($\mathbb{R}^2$) cylindrical, spherical ($\mathbb{R}^3$)
- Int, Ext, ∂, open, closed, boundedness, path-connect.
  (un)

---

multi-variable vector valued function.

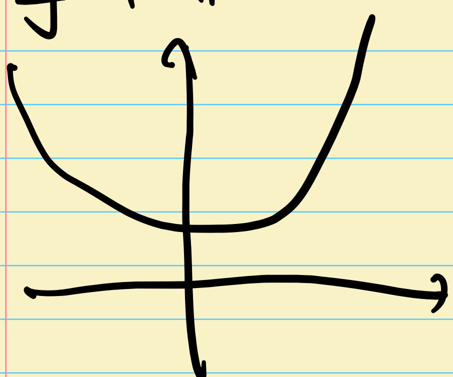
$$\vec{f} : \Omega \; (\subset \mathbb{R}^n) \longrightarrow \mathbb{R}^m$$

Try to visualize $\vec{f}$

① Graph of $\vec{f}$

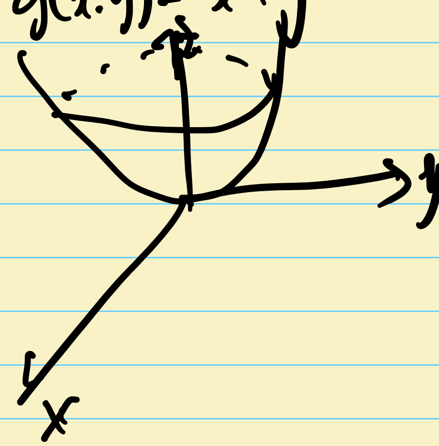
$$\text{Graph}(\vec{f}) = \{ (\underset{\mathbb{R}^n}{\vec{x}}, \underset{\mathbb{R}^m}{\vec{f}(\vec{x})}) \in \mathbb{R}^{n+m} \mid \vec{x} \in \Omega \} \subset \mathbb{R}^{n+m}$$

eg $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 1 + x^2$



$g : \mathbb{R}^2 \to \mathbb{R}$

$g(x,y) = x^2 + y^2$



$h : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$

$h(x,y,z) = (x^2+y^2, y^2+z^2)$

$\text{Graph}(h) \subset \mathbb{R}^3 \times \mathbb{R}^2 = \mathbb{R}^5$ hard to draw.

② Level set of $\vec{f}: \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^m$.

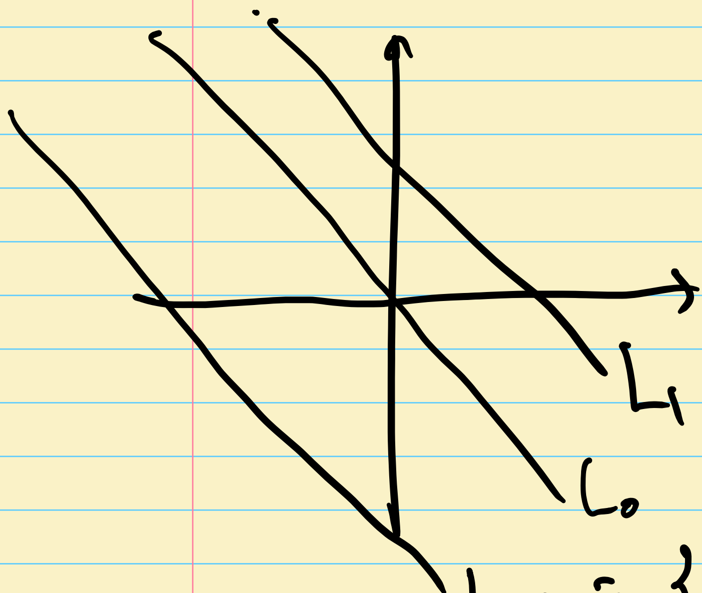
If $c \in \mathbb{R}^m$, define the level set of c to be

$$L_c = \{\vec{x} \in \Omega \mid \vec{f}(\vec{x}) = c\}$$

$$= \vec{f}^{-1}(c) \subseteq \Omega \subseteq \mathbb{R}^n.$$

eg $f: \mathbb{R}^2 \to \mathbb{R}$ $\quad f(x,y) = x+y$

$L_0 = \{f(x,y) = 0\} = \{x+y=0\}$



$L_1 = \{x+y=1\}$

L_0

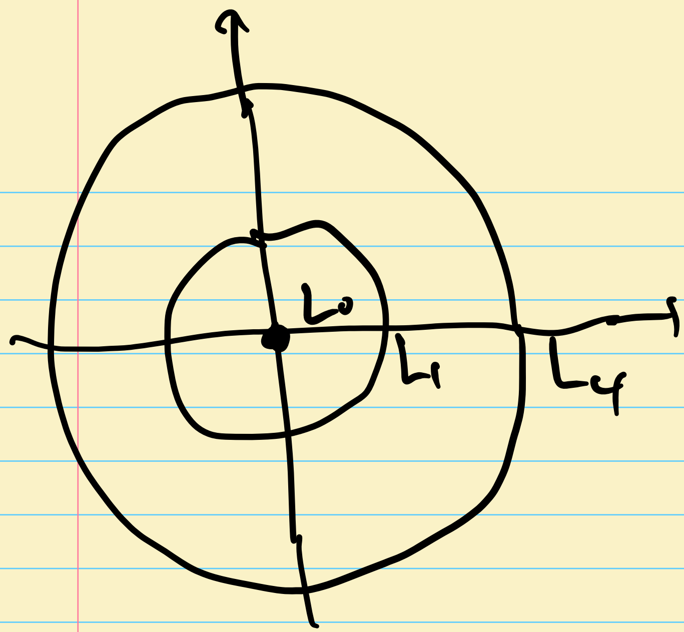
$L_{-2} = \{x+y=-2\}$

eg $g: \mathbb{R}^2 \to \mathbb{R}$ $\quad g(x,y) = x^2+y^2$

If $c<0$, $L_c = \{x^2+y^2=c\} = \emptyset$

If $c=0$, $L_c = \{(0,0)\}$ a point

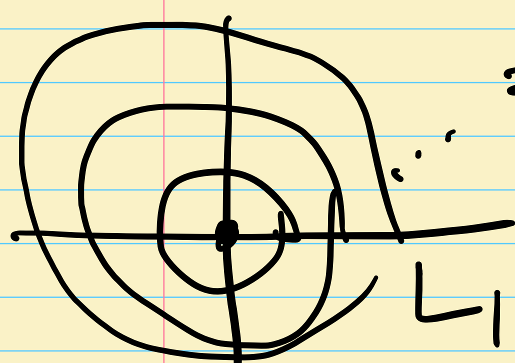
If $c>0$, $L_c = \{x^2+y^2=c\}$ circle of radius $\sqrt{c}$.



eg $h : \mathbb{R}^2 \to \mathbb{R}$ $\quad h(x,y) = \cos(2\pi(x^2+y^2))$

L_1 ? $\quad h(x,y) = 1 \iff x^2+y^2 \in \mathbb{Z}$

$L_1 = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \in \mathbb{Z}\}$



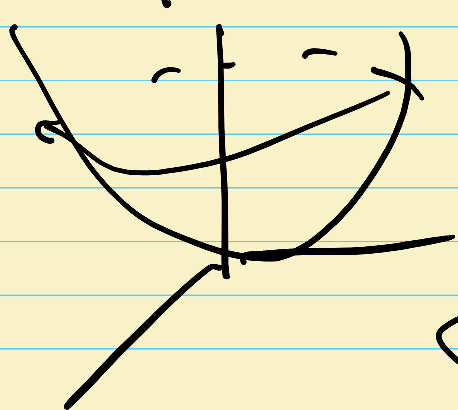
= origin $\cup$ infinitely many circles with radius $\sqrt{k}$ ($k \in \mathbb{Z}$)

<u>Level set and graph</u>

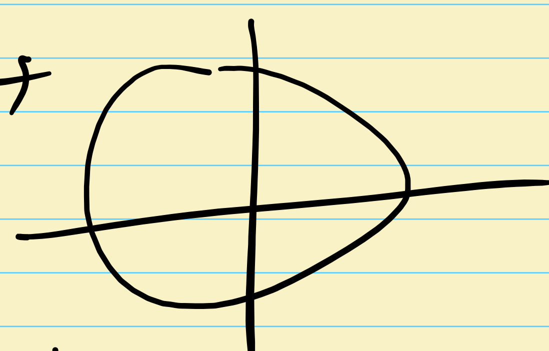
$f : \mathbb{R}^2 \to \mathbb{R}$

Graph$(f) \subset \mathbb{R}^3$

level set of $f \subset \mathbb{R}^2$



taking slice w.r.t. $z = c$.



Limit of multi-variable functions

$f: \mathbb{R} \to \mathbb{R}$ "$\lim_{x \to a} f(x) = L$"

Let $A \subseteq \mathbb{R}^n$ be a subset.
Define the closure of A. $\overline{A} = A \cup \partial A$.
We want to define "as $\vec{x}$ very closed to $\vec{a} \in \overline{A}$, $f(\vec{x})$ very close to $\vec{L}$".

Def We say that the limit of $\vec{f}$ at $\vec{a}$ is $\vec{L}$, denoted by $\lim_{\vec{x} \to \vec{a}} \vec{f}(\vec{x}) = \vec{L}$, if

for any $\varepsilon > 0$, there exists a $\delta > 0$ s.t. $\|\vec{f}(\vec{x}) - \vec{L}\| < \varepsilon$ for any $\vec{x} \in A$ satisfying $0 < \|\vec{x} - \vec{a}\| < \delta$.

Rmk $0 < \|\vec{x} - \vec{a}\|$: $\vec{x} = \vec{a}$ is not included.

Example $f: \mathbb{R}^2 \to \mathbb{R}$ $f(x,y) = x+y$.
Then $\lim_{(x,y) \to (1,2)} f(x,y) = 3$.

(proof) For example, $\varepsilon = 1$, $\rightsquigarrow \delta$! one can pick $\delta = 1/2$

we need to check

For any $0 < \|\vec{x} - (1,2)\| < \frac{1}{2}$, $\| x+y-3 \| < 1$.

$$|x-1| = \sqrt{(x-1)^2} \leq \sqrt{(x-1)^2 + (y-2)^2} < \frac{1}{2}$$

$$|y-2| = \sqrt{(y-2)^2} \leq \sqrt{(x-1)^2 + (y-2)^2} < \frac{1}{2}$$

$$\begin{aligned} \|x+y-3\| &= \|(x-1)+(y-2)\| \\ &\leq \|x-1\| + \|y-2\| \\ &< \frac{1}{2} + \frac{1}{2} \\ &= 1 = \varepsilon \end{aligned}$$

What about $\varepsilon = \frac{1}{100}$! one can pick $\delta = \frac{1}{200}$

For any $\varepsilon > 0$, one can pick $\delta = \frac{\varepsilon}{2}$.

we need to check

; for any $\vec{x}$ satisfying $\|\vec{x} - (1,2)\| < \delta = \frac{\varepsilon}{2}$,

$\|f(\vec{x}) - 3\| < \varepsilon$.

If $\|\vec{x} - (1,2)\| < \delta = \frac{\varepsilon}{2}$,

the $\sqrt{(x-1)^2 + (y-2)^2} < \frac{\varepsilon}{2}$

$$|x-1| = \sqrt{(x-1)^2} \leq \sqrt{(x-1)^2+(y-2)^2} < \frac{\varepsilon}{2}$$

$$|y-2| = \sqrt{(y-2)^2} \leq \quad " \quad < \frac{\varepsilon}{2}$$

$$\therefore \ |f(\vec{x}) - 3| = |x+y-3|$$

$$= |(x-1)+(y-2)|$$

$$\leq |x-1| + |y-2|$$

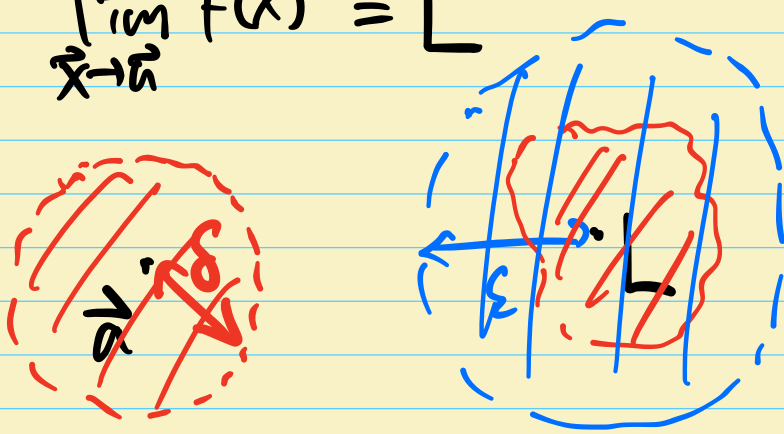
$$< \varepsilon$$

$$\therefore \lim_{(x,y)\to(1,2)} f(\vec{x}) = 3. \quad \square$$

Picture

$$A (\subseteq \mathbb{R}^n) \xrightarrow{f} \mathbb{R}^m$$

$$\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = L$$



Example $f(x,y) = x^2+y^2$. $\lim_{(x,y)\to(0,0)} f(x,y) = 0$.

(sol) Need show that :

for any $\varepsilon>0$, there is a δ s.t.

$$\underbrace{0 < \|(x,y) - (0,0)\| < \delta}_{} \Rightarrow \|f(x,y) - 0\| < \varepsilon.$$

$$0 < \sqrt{x^2+y^2} < \delta \Rightarrow |x^2+y^2| < \varepsilon$$

We may $\delta = \sqrt{\varepsilon}$ (or smaller)

$$\sqrt{x^2+y^2} < \delta \Rightarrow |x^2+y^2| < \varepsilon.$$

Exercise complete the proof following this idea.

$\square$

Let $A \subseteq \mathbb{R}^n$, $\vec{a} \in \overline{A}$, $\vec{f} : A \to \mathbb{R}^m$

$$\vec{f}(\vec{x}) = \begin{pmatrix} f_1(\vec{x}) \\ \vdots \\ f_m(\vec{x}) \end{pmatrix} \qquad f_i : A \to \mathbb{R}$$

Prop $\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = L = \begin{pmatrix} l_1 \\ \vdots \\ l_m \end{pmatrix} \iff \lim_{\vec{x}\to\vec{a}} f_i(\vec{x}) = l_i$

i.e. enough to focus on limit of real-valued functions $f: A \to \mathbb{R}$

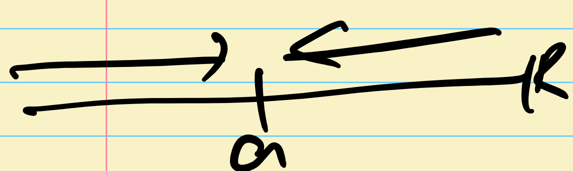
Eg $f: \mathbb{R}^2 \to \mathbb{R}^2$

$$f(x,y) = \begin{pmatrix} x+y \\ x^2+y^2+1 \end{pmatrix}$$

$$\lim_{(x,y)\to(1,2)} f(x,y) = \begin{pmatrix} \lim_{(x,y)\to(1,2)} x+y \\ \lim_{(x,y)\to(1,2)} x^2+y^2+1 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$$

Limit along a path

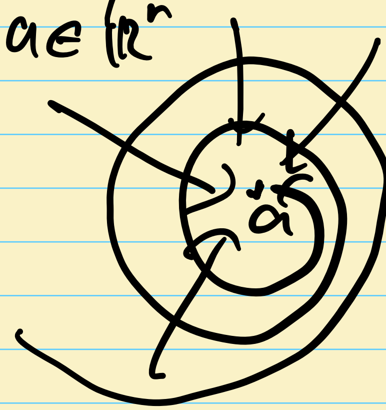
one variable



essentially two ways to approach to a.

n variable $n \geq 2$

many ways to approach $\vec{a} \in \mathbb{R}^n$



Fact $f: A \subseteq \mathbb{R}^n \to \mathbb{R}$, $\vec{a} \in \overline{A}$

$\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = L \iff$ limit of $f(\vec{x})$ along any path exists and equals to L.

This fact is useless for computing / proving existence of L.

but useful for showing that limit does not exist.

$\therefore$ · Find one path s.t. the limit along that path does not exist

or · Find two paths s.t. the limits along that two paths are different

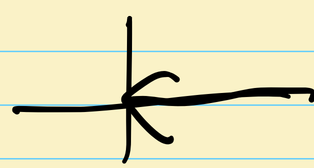
$\Rightarrow$ $\lim_{\vec{x}\to\vec{a}} f(\vec{x})$ does not exist.

<u>Example</u> $f(x,y)=\dfrac{x^2-y^2}{x^2+y^2}$: $\underbrace{\mathbb{R}^2\setminus\{0\}}_{\Omega} \longrightarrow \mathbb{R}^2$ $\quad \bar{\Omega}=\mathbb{R}^2$

Note that f is not defined at $\{(0,0)\}$.

$\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

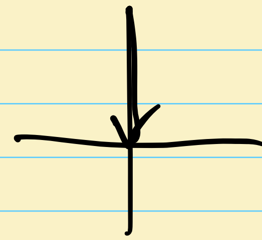
(sol) ① Along x-axis;



i.e. $\lim_{x\to 0} f(x,0)$

$$= \lim_{x \to 0} \frac{x^2 - 0^2}{x^2 + 0^2} = \lim_{x \to 0} 1 = 1.$$

② Along y-axis



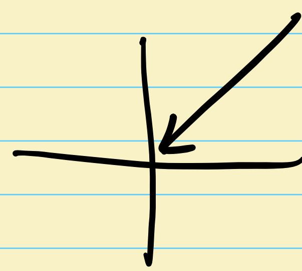
: $\lim_{y \to 0} f(0, y)$

$$= \lim_{y \to 0} \frac{0^2 - y^2}{0^2 + y^2} = \lim_{y \to 0} -1 = -1.$$

These two limits are different, hence

$\lim_{(x,y) \to (0,0)} f(x, y)$ does not exist.

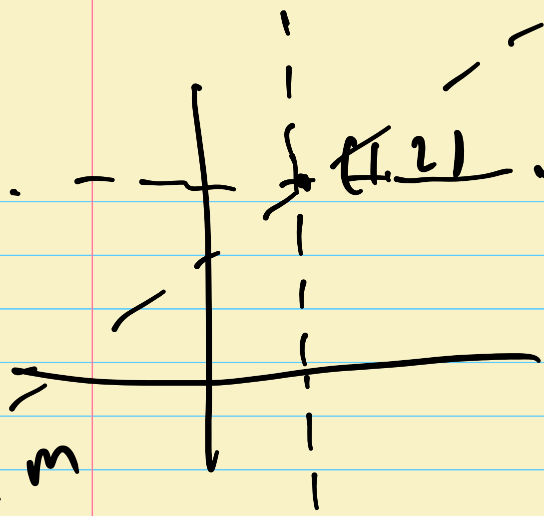
Rmk Along $y = x$



$$\lim_{\substack{(x,y) \to (0,0) \\ y = x}} \frac{x^2 - x^2}{x^2 + x^2} = \lim 0 = 0$$

Example $\lim_{(x,y) \to (1,2)} \frac{(x-1)(y-2)}{(x-1)^2 + (y-2)^2}$ exist?

(Sol)



① Along $x=1$.

$$\lim_{\substack{(x,y)\to(1,2)\\ x=1}} \frac{(x-1)(y-2)}{(x-1)^2+(y-2)^2}$$

$$= \lim_{\substack{(x,y)\to(1,2)\\ x=1}} 0 \qquad = 0.$$

② Along $y=2$.

$$\lim_{\substack{(x,y)\to(1,2)\\ y=2}} \frac{(x-1)(y-2)}{(x-1)^2+(y-2)^2} = \lim 0 = 0.$$

③ Along $y-2=m(x-1)$: a line passing from $(1,2)$ of slope m.

$$\lim_{\substack{(x,y)\to(1,2)\\ y-2=m(x-1)}} \frac{(x-1)(y-2)}{(x-1)^2+(y-2)^2} = \lim \frac{m(x-1)^2}{(x-1)^2+m^2(x-1)^2}$$

$$= \lim \frac{m}{1+m^2}$$

$$= \frac{m}{1+m^2}$$

$\frac{m}{1+m^2}$ has different values for different m.

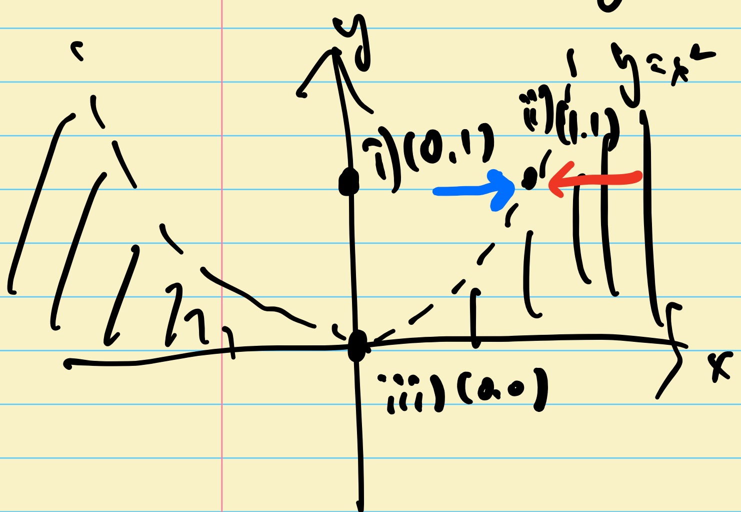
eg $m=1$; $\frac{m}{1+m^2}=\frac{1}{2}$

$m=0$; $\frac{m}{1+m^2}=0$.

$\therefore \lim_{(x,y)\to(1,2)} \frac{(x-1)(y-2)}{(x-1)^2+(y-2)^2}$ does not exist.

Example $f: \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} 1 & \text{if } 0 < y < x^2 \\ 0 & \text{otherwise.} \end{cases}$$



$f=1$ on ////

$f=0$ on other

i) $\lim_{(x,y)\to(0,1)} f(x,y) =$

near $(0,1)$, $f=0$ hence

$\lim_{(x,y)\to(0,1)} f(x,y) = 0$.

ii) $\lim\limits_{(x,y)\to(1,1)} f(x,y) = ?$

$$\lim_{\substack{(x,y)\to(1,1)\\ y=1,\, x>1}} f(x,y) = \lim 1 = 1$$

$$\lim_{\substack{(x,y)\to(1,1)\\ y=1,\, x<1}} f(x,y) = \lim 0 = 0$$

$\therefore \lim\limits_{(x,y)\to(1,1)} f(x,y)$ does not exist.

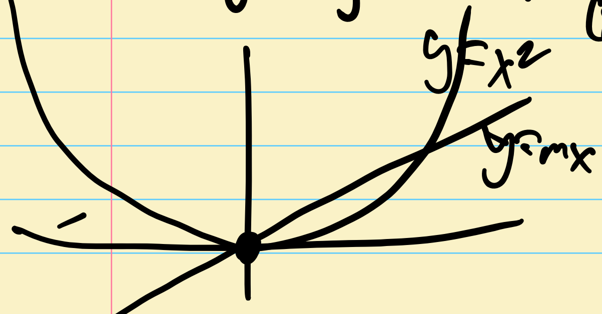
iii) $\lim\limits_{(x,y)\to(0,0)} f(x,y)$?

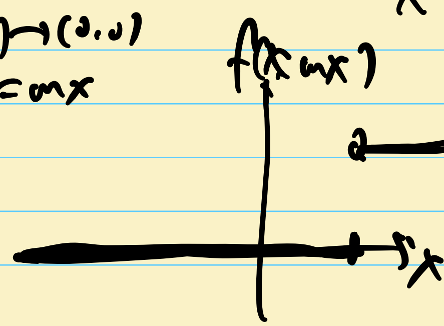
Along y-axis; $f=0$ on y-axis

$$\therefore \lim_{\substack{(x,y)\to(0,0)\\ x=0}} f(x,y) = 0$$

Along $y=mx$ $(m>0)$; $\lim\limits_{\substack{(x,y)\to(0,0)\\ y=mx}} f(x,y) = \lim\limits_{x\to 0} f(x,mx)$

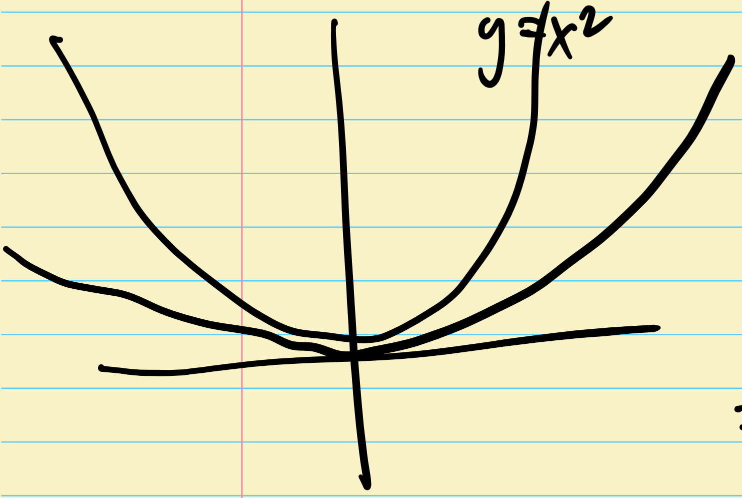
$= 0$





Similar for $m=0$, $m<0$.

Along the curve $y=\frac{1}{2}x^2$



$$\lim_{\substack{(x,y)\to(0,0)\\ y=\frac{1}{2}x^2}} f(x,y)$$

$$= \lim_{x\to 0} f(x,\tfrac{1}{2}x^2)$$

$$\left(f(x,\tfrac{1}{2}x^2) = \begin{cases} 1 & x\neq 0 \\ 0 & x=0 \end{cases} \right)$$

$$= 1$$

not equal to the limit along y-axis or $y=mx$

$\therefore \lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

(Exercise $f(x) = \begin{cases} 0 & x\neq 0 \\ 1 & x=0 \end{cases}$ $\quad \lim_{x\to 0} f(x) = 0 \neq 1$)

Properties of limits.

Prop Assume all limits on the right hand side exist. Then the limit on the left hand side exists and the formula holds.

① $\lim_{\vec{x}\to\vec{a}} f(\vec{x}) \pm g(\vec{x}) = \lim_{\vec{x}\to\vec{a}} f(\vec{x}) \pm \lim_{\vec{x}\to\vec{a}} g(\vec{x})$

② $\lim_{\vec{x}\to\vec{a}} k f(\vec{x}) = k \lim_{\vec{x}\to\vec{a}} f(\vec{x})$ where k is a constant

③ $\lim_{\vec{x}\to\vec{a}} f(\vec{x}) g(\vec{x}) = \lim_{\vec{x}\to\vec{a}} f(\vec{x}) \lim_{\vec{x}\to\vec{a}} g(\vec{x})$

④ $\lim_{\vec{x}\to\vec{a}} \frac{f(\vec{x})}{g(\vec{x})} = \frac{\lim_{\vec{x}\to\vec{a}} f(\vec{x})}{\lim_{\vec{x}\to\vec{a}} g(\vec{x})}$ if $\lim_{\vec{x}\to\vec{a}} g(\vec{x}) \neq 0$

⑤ $\lim_{\vec{x}\to\vec{a}} f(\vec{x})^n = \left(\lim_{\vec{x}\to\vec{a}} f(\vec{x})\right)^n$, $n \geq 0$

⑥ $\lim_{\vec{x}\to\vec{a}} f(\vec{x})^{\frac{1}{n}} = \left(\lim_{\vec{x}\to\vec{a}} f(\vec{x})\right)^{\frac{1}{n}}$

(If n is even, assume $f(\vec{x}) \geq 0$ near $\vec{a}$).

eg $\displaystyle\lim_{(x,y)\to(1,2)} x^2 + 2xy + 3y^2$

$$\left.\begin{array}{l}\displaystyle\lim_{(x,y)\to(1,2)} x = 1 \\[2em] \displaystyle\lim_{(x,y)\to(1,2)} y = 2\end{array}\right] \longrightarrow \begin{array}{l}\lim x^2 = (\lim x)^2 = 1 \\ \lim 2xy = 2(\lim x)(\lim y) = 4 \\ \lim 3y^2 = 3(\lim y)^2 = 3 \cdot 2^2 = 12\end{array}$$

$$\Rightarrow \lim_{\substack{(x,y)\\ \to(1,2)}} x^2 + 2xy + 3y^2 = 17.$$

Thm (Sandwich theorem, squeeze theorem)

Let $f, g, h : \Omega\ (\subseteq \mathbb{R}^n) \to \mathbb{R}$.

If $g(\vec{x}) \leq f(\vec{x}) \leq h(\vec{x})$ near $\vec{a} \in \Omega$.

and $\displaystyle\lim_{\vec{x}\to\vec{a}} g(\vec{x}) = \lim_{\vec{x}\to\vec{a}} h(\vec{x}) = L$,

then $\displaystyle\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = L$.

In particular, if $|f(\vec{x})| \leq g(\vec{x})$

$\Rightarrow -g(\vec{x}) \leq f(\vec{x}) \leq g(\vec{x})$

If $|f(\vec{x})| \le g(\vec{x})$, $\lim_{\vec{x}\to\vec{a}} g(\vec{x}) = 0$

(hence $\lim_{\vec{x}\to\vec{a}} -g(\vec{x}) = 0$)

then $\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = 0$.

<u>Example</u>

$$\lim_{(x,y)\to(0,0)} x\cdot\cos\left(\frac{1}{x^2+y^2}\right) = 0$$

(sol)

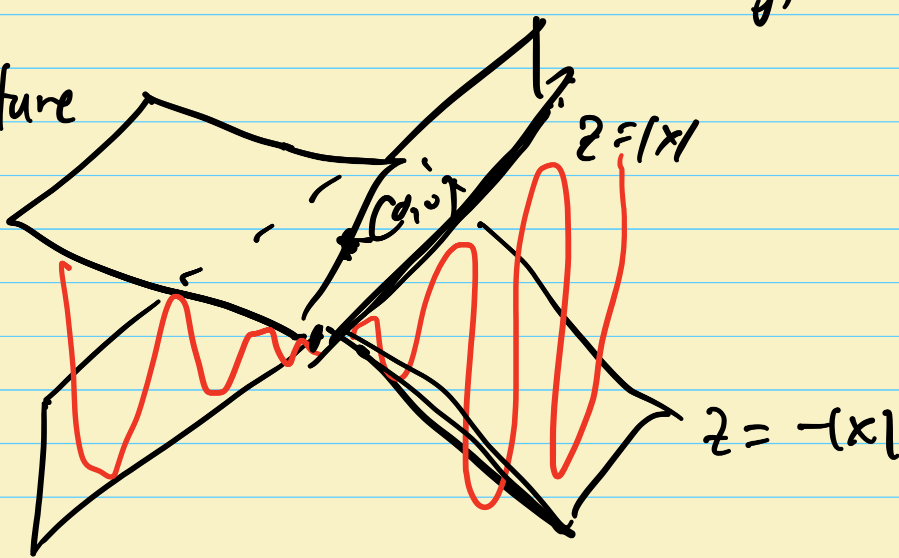
$$\left|\cos\left(\frac{1}{x^2+y^2}\right)\right| \le 1$$

$$\left|x\cos\left(\frac{1}{x^2+y^2}\right)\right| \le |x|$$

Also, $\lim_{(x,y)\to\vec{0}} |x| = 0$.

By the squeeze theorem, $\lim_{(x,y)\to(0,0)} x\cos\left(\frac{1}{x^2+y^2}\right) = 0$.

Picture



Example $\displaystyle \lim_{(x,y)\to(1,0)} \frac{(x-1)^2 \ln x}{(x-1)^2+y^2}$

(sol)

$$\left| \frac{(x-1)^2 \ln x}{(x-1)^2+y^2} \right| = \left| \frac{(x-1)^2}{(x-1)^2+y^2} \right| |\ln x|$$

$$\leq |\ln x| .$$

Also, $\displaystyle \lim_{(x,y)\to(1,0)} |\ln x| = |\ln 1| = 0$

By squeeze theorem,

$$\lim_{(x,y)\to(1,0)} \frac{(x-1)^2 \ln x}{(x-1)^2+y^2} = 0.$$